Closest point between two rays

http://palitri.com

1 Quick solution

Given a ray with initial point A, direction $\vec{a}$ and a ray with initial point B, direction $\vec{b}$, the points at which each vector is at closest proximity to the other are

$$D = A + \vec{a}\frac{-\left(\vec{a}\cdot\vec{b}\right)\left(\vec{b}\cdot\vec{c}\right) + \left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{b}\right)}{\left(\vec{a}\cdot\vec{a}\right)\left(\vec{b}\cdot\vec{b}\right) - \left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{b}\right)}$$

$$E = B + \vec{b}\frac{\left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{c}\right) - \left(\vec{b}\cdot\vec{c}\right)\left(\vec{a}\cdot\vec{a}\right)}{\left(\vec{a}\cdot\vec{a}\right)\left(\vec{b}\cdot\vec{b}\right) - \left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{b}\right)}$$

where D is the point along $\vec{a}$, E is the point along $\vec{b}$ and $\vec{c} = \vec{AB}$.
The point, which is closest simultaneously to both rays and is at the same distance between them can be expressed as

$$\frac{D + E}{2}$$

and the shortest distance between the rays is

$$\left|\vec{DE}\right|$$

For consistency, the following conditions should be met

$$\vec{a}\cdot\vec{a} \neq 0$$

$$\vec{b}\cdot\vec{b} \neq 0$$

$$\left(\vec{a}\cdot\vec{b}\right)^2 \neq \left(\vec{a}\cdot\vec{a}\right)\left(\vec{b}\cdot\vec{b}\right)$$

See fig. Concept for reference.
The solution is thoroughly covered in the rest of the document.

2 Introduction

In this document we'll calculate

- Point on a ray, that is closest to another ray
- Closest point between two rays
- Shortest distance between two rays

Note that because we'll be looking for points of proximity with specific locations, this solution will not work with collinear rays, for which these points would be ambiguous, as in such case all points along any of the rays would be at the same distance from the other ray.

3 Setting

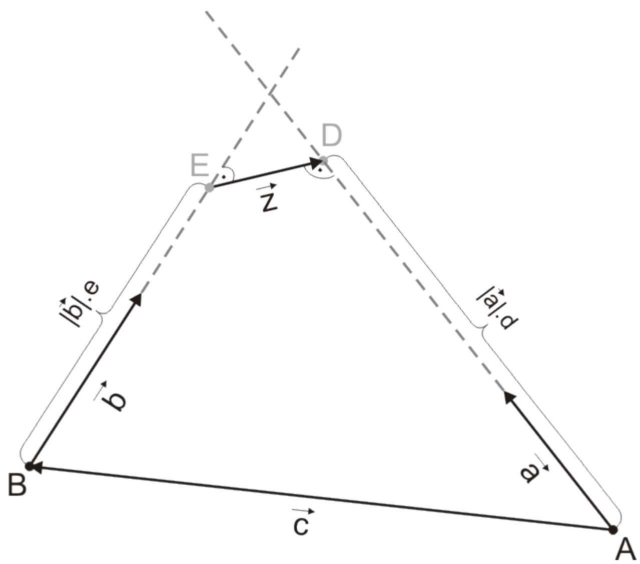

Figure 1: Concept

We have two rays, each having an initial point in space given by its coordinates and a direction, given by a vector.

- Ray with starting point A and direction, given by vector $\vec{a}$
- Ray with starting point B and direction, given by vector $\vec{b}$

We'll calculate the exact point on each ray, at which it gets closest to the other ray. The distance between these points is the distance between the rays and the middle of the segment, defined by these points will give us the closest point between both rays.
Also, if the distance is zero, then the two points coincide and the the rays intersect.

4 Solution

Suppose point D, lying along $\vec{a}$, is the closest to $\vec{b}$.
We can express $\vec{AD}$ as $\vec{a}$, multiplied by some scalar factor, let's name this factor d. This way

$$\vec{AD} = \vec{a}d$$

Analogically, suppose point E, lying along $\vec{b}$, is the closest to $\vec{a}$ and

$$\vec{BE} = \vec{b}e$$

Finding the two scalar values d and e will solve the problem.

For convenience, let's have

$$\vec{c} = \vec{AB}$$

$$\vec{z} = \vec{ED}$$

Now, let's see what we have.
The shortest segment between two lines in a three dimensional space is perpendicular to both lines and only one such segment exists if the lines are not parallel. In our case, this means that $\vec{z}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
Also, obviously, $\vec{c} + \vec{b}e + \vec{z} - \vec{a}d = \vec{0}$.

So, let's write it all down

$$\vec{a} \cdot \vec{z} = 0$$

$$\vec{b} \cdot \vec{z} = 0$$

$$\vec{c} + \vec{b}e + \vec{z} - \vec{a}d = \vec{0}$$

We've got three unknowns - d, e, $\vec{z}$ and three equations, so we can solve this equation set and get d and e.

From the third equation, we can express $\vec{z} = \vec{a}d - \vec{b}e - \vec{c}$ and get rid of $\vec{z}$ by replacing it in the first two equations with this expression.

$$\vec{a} \cdot \vec{a}d - \vec{a} \cdot \vec{b}e - \vec{a} \cdot \vec{c} = 0$$

$$\vec{a} \cdot \vec{b}d - \vec{b} \cdot \vec{b}e - \vec{b} \cdot \vec{c} = 0$$

For convenience, let's substitute all the dot products

$$p = \vec{a} \cdot \vec{b}$$

$$q = \vec{a} \cdot \vec{c}$$

$$r = \vec{b} \cdot \vec{c}$$

$$s = \vec{a} \cdot \vec{a}$$

$$t = \vec{b} \cdot \vec{b}$$

With this, our equation set looks much simpler

$$\begin{aligned} sd - pe - q &= 0 \\ pd - te - r &= 0 \end{aligned}$$

Now, express d from the first equation and e from the second

$$\begin{aligned} d &= \frac{pe+q}{s}, s \neq 0 \\ e &= \frac{pd-r}{t}, t \neq 0 \end{aligned}$$

Replace e in the first equation by its expression from the second and replace d in the second equation by its expression from the first.

$$\begin{aligned} d &= \frac{p^2 d - pr}{st} + \frac{q}{s} \\ e &= \frac{p^2 e + pq}{st} - \frac{r}{t} \end{aligned}$$

Leave the right side of each equation with no variables

$$\begin{aligned} d\left(1 - \frac{p^2}{st}\right) &= -\frac{pr}{st} + \frac{q}{s} \\ e\left(1 - \frac{p^2}{st}\right) &= \frac{pq}{st} - \frac{r}{t} \end{aligned}$$

Leave the left side of each equation with its variable only

$$d = -\frac{pr}{st - p^2} + \frac{q}{s - \frac{p^2}{t}}, p^2 \neq st$$

$$e = \frac{pq}{st - p^2} - \frac{r}{t - \frac{p^2}{s}}$$

And finally, we have

$$\begin{aligned} d &= \frac{-pr + qt}{st - p^2} \\ e &= \frac{pq - rs}{st - p^2} \end{aligned}$$

With the restrictions

$$\begin{aligned} s &\neq 0 \\ t &\neq 0 \\ p^2 &\neq st \end{aligned}$$

5 Notes

5.1 Restrictions

Looking back at the substitutions we did, let's take a brief to see what Math is saying with the restrictions that popped out while solving the equation set.

- $s \neq 0$, or $\vec{a} \cdot \vec{a} \neq 0$. That is $|\vec{a}|\,|\vec{a}| \cos\angle(\vec{a}, \vec{a}) \neq 0$. With the cosine being equal to 1, this leads to $|\vec{a}| \neq 0$. And of course, if the length of $\vec{a}$ is zero, it would be a point rather than a vector and our very setting would be inconsistent.
- $t \neq 0$ is analogous to $s \neq 0$ and means that $\left|\vec{b}\right| \neq 0$
- $p^2 \neq st$, or $\left(\vec{a} \cdot \vec{b}\right)^2 \neq (\vec{a} \cdot \vec{a}) \left(\vec{b} \cdot \vec{b}\right)$. That is $|\vec{a}| \left|\vec{b}\right| \cos\angle\left(\vec{a}, \vec{b}\right) |\vec{a}| \left|\vec{b}\right| \cos\angle\left(\vec{a}, \vec{b}\right) \neq |\vec{a}|\,|\vec{a}|\, 1 \left|\vec{b}\right| \left|\vec{b}\right| 1$, which leaves $\cos\angle\left(\vec{a}, \vec{b}\right) \neq 1$. In other words, it says $\vec{a}$ and $\vec{b}$ should not be collinear.

  This restriction appears, because of the way we've formulate our very question - essentially, we are looking for a specific point on each ray, which is closest to the other ray. But if both rays are collinear, all points along the rays would satisfy our request and thus there wouldn't be a single, specific answer.

  Nevertheless, even in such a case, THERE IS A SPECIFIC DISTANCE. And if we are looking for the distance, rather than any specific point, it would be best to formulate our mathematical request in the respective a manner.

5.2 Closest point between both rays

As mentioned, the point, which is closest simultaneously to both rays is at equal distance from them and is the middle of the segment DE. We can easily find it as

$$\frac{D+E}{2}$$

or, with D and E expressed

$$\frac{A + \vec{a}d + B + \vec{b}e}{2}$$

5.3 Distance between rays

The distance between the two rays is the distance between points D and E, that is, the length of $\vec{z}$.

5.4 Distance between initial and closest point

We can easily calculate the distance between the initial point of a ray and the point, closest to the other ray. It's simply
$\frac{d}{|\vec{a}|}$ for $\vec{a}$ and
$\frac{e}{|\vec{b}|}$ for $\vec{b}$
And if the ray vectors are normalized, this leaves only d or e respectively.